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Multiple trigonometrical sums according to Arhipov, Karacuba and Čubarikov

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As a topic in the theory of exponential sums we shall present here without proof the outlines of the theory of multiple trigonometrical sums following the monograph [2]. The reason why I have chosen just this one article is that the theory has been got so well into shape and expounded in detail in the monograph that one can do without other enormous papers by the authors themselves to go through further developements (see, e.g. [3], [4]). I will try to follow and reproduce the exposition of the monograph as faithfully as possible so that the reader may get the information before it is translated as Trudy Mat. Inst. Steklov 151. For example, the bold-face figures that will appear stand for the numbers of the corresponding paragraphs in [2].

First we fix some notation. Let

$$F(x_1,\ldots,x_r) = F_A(x_1,\ldots,x_r) = \sum_{t_1=0}^{n_1} \cdots \sum_{t_r=0}^{n_r} \alpha(t_1,\ldots,t_r)x_1^{t_1}\cdots x_r^{t_r}$$

$$\in \mathbb{R}[x_1,\ldots,x_r],$$

where $A = (\alpha(0,\ldots,0),\alpha(0,0,\ldots,1),\ldots,\alpha(n_1,\ldots,n_r))$ is an m-dimensional vector, m being given by

$$m = \prod_{i=1}^{r} (n_i + 1).$$

Define the r-multiple trigonometrical sum (abbreviated: m.t.s) $S_t(A)$ by

$$S_t(A) = \sum_{x_1=1}^{P_1} \cdots \sum_{x_r=1}^{P_r} \exp(2\pi i t F(x_1,\ldots,x_r))$$

where

$$P \geq P_i \in \mathbb{N},\ i = 1,\ldots,r, \qquad t \in \mathbb{R}.$$

Further, we write

$$S = S(A) = S_1(A),$$

and with the estimation of which we shall mainly engage ourselves. Since S is a periodic function of $\alpha(t_1,\ldots,t_r)$, enough to estimate it for $\alpha(t_1,\ldots,t_r) \in \Omega$, the m-dimensional unit cube, i.e.

$$\Omega : \quad 0 \leq \alpha(t_1,\ldots,t_r) \leq 1,\ 0 \leq t_1 \leq n_1,\ldots,\ 0 \leq t_r \leq n_r .$$

When $r = 1$, the sum

$$S = \sum_{x=1}^{P} \exp(2\pi i f(x)), \quad f(x) = \alpha_1 x + \cdots + \alpha_n x^n$$

carries the name of Weyl's exponential sum.

In the monograph the case

$$P_1 = \cdots = P_r = P, \quad n_1 = \cdots = n_r = n$$

is considered. The general case in which the values of main parameters P_i $(i = 1,\ldots,r)$ are essentially different in their order of magnitude is stated as Problem B in Conclusion. This problem has recently been solved out by the authors in their papers [3], [4].

For the estimation of $S(A)$ we need the estimate of the so-called Vinogradov's integral $J = J(P;n,k,r)$ defined by

$$J = \int \cdots \int_{\Omega} |S(A)|^{2k}\, dA = \underbrace{\int_0^1 \cdots \int_0^1}_{m} |S(A)|^{2k} d\alpha(0,\ldots,0)\cdots d\alpha(n,\ldots,n),$$

$$J(P;n,k) := J(P;n,k,1).$$

Clearly, $J(P;n,k,r)$ is the number of solutions of the following system of Diophantine eqns:

$$\sum_{j=1}^{2k} (-1)^j\, x_{1,j}^{t_1} \cdots x_{r,j}^{t_r} = 0, \quad 0 \le t_1,\ldots,t_r \le n, \quad 1 \le x_{1,j},\ldots,x_{r,j} \le P,$$
$$j = 1,\ldots,2k,$$

and $J(P;n,k)$ is the number of solutions of the following Diophantine eqns (in the case of single sum this has the name of Vinogradov's integral):

$$\begin{cases} x_1 + \cdots + x_k = y_1 + \cdots + y_k, \\ x_1^2 + \cdots + x_k^2 = y_1^2 + \cdots + y_k^2, \\ \cdots\cdots \\ x_1^n + \cdots + x_k^n = y_1^n + \cdots + y_k^n, \end{cases}$$

$$1 \le x_1,\ldots,x_k,y_1,\ldots,y_k \le P.$$

Here J has the following expression as an integral:

$$J = \int_0^1 \cdots \int_0^1 \left| \sum_{x \le P} \exp(2\pi i f(x)) \right|^{2k} d\alpha_1 \cdots d\alpha_n .$$

Chap. I Mean value theorems

§1. p-adic proof of Vinogradov's mean value theorem (i.e. the estimate of Vinogradov's integral J) is given, which helps one to go through the complicated proof of the mean value theorem on m.t.s.

4. Fundamental theorem. Let $\tau, r_1, \ldots, r_\tau, n, k \in \mathbf{N}$,

$$1 = r_1 \le r_2 \le \cdots \le r_\tau \le n, \quad \Delta(\tau) = \left(n - \frac{r_\tau - 1}{2}\right) + \left(1 - \frac{1}{r_\tau}\right)$$

$$\cdot\left(n - \frac{r_{\tau-1} - 1}{2}\right) + \cdots + \left(1 - \frac{1}{r_\tau}\right)\left(1 - \frac{1}{r_{\tau-1}}\right)\cdots\left(1 - \frac{1}{r_2}\right)\left(n - \frac{r_1 - 1}{2}\right),$$

$\kappa_\tau = \sum_{j=1}^{\tau} (r_j^2 + \Delta(j))$. Then for $k \ge n\tau$,

$$J \le n^{2\Delta(\tau) r_\tau}\, 2^{\kappa_\tau} (8k)^{2n\tau}\, P^{2k-\Delta(\tau)} .$$

Corollary to Theorem 1. For $\frac{1}{2} > \varepsilon > 0$, if $k = mn \le \varepsilon^2 n^2$, then

$$J \ll P^{k(1+\varepsilon)} ,$$

which is almost precise, since always $J \gg P^k$.

From the Fundamental theorem it does not follow that when $k \asymp n^2 \log n$, we have

$$J \ll P^{2k - \frac{n(n+1)}{2}} \qquad \text{(Vinogradov)},$$

the <u>simplified bound</u> for J, but Theorem 2 enables one to obtain estimates for J when k is near to $\frac{n(n+1)}{2}$, the case useful for applications.

Vinogradov's mean value theorem is proved by a p-aidc method whose basis is the following <u>fundamental recurrence inequality</u>:

$$J \le aTJ_1 + bJ_2, \qquad (4)$$

where a and b are quantities explicitly given, T the number of solutions of a system of congruences, J_1 and J_2 are quantities of the same nature as J, however, with smaller values of parameters. (4) is simplified here (with $J_2 = 1$).

3. Lemma 3. If $k \ge n$, $r \in \mathbf{N}$, $P \ge 1$, $r \le n$, then $\exists p \in [P^{1/r}, 2P^{1/r}]$ such that

$$J \le 2r^2 \binom{k}{n}^2 p^{2k-2n} TJ(P_1; n.k-n) + (2n)^{2kr} P^{k-1}$$

$$\le 4k^{2n}\, p^{2k-2n+\frac{r(r-1)}{2}}\, P^n\, J(P_1;,n,k-n) + (2n)^{2kr}\, P^k, \qquad (11)$$

where $P_1 = Pp^{-1} + 1$.

§2. Theorem on the mean value of the 2k-th power of the modulus of r-m.t.s.

We say that a vector $\bar{x}_j = \{(x_{1,j},\dots,x_{r,j}),\ 1 \le j \le k\}$ is <u>regular</u> iff

$$\text{rank } M = \text{rank}\,(x_{1,j}^{t_1},\dots,x_{r,j}^{t_r})_{\substack{0\le t_1,\dots,t_r\le n\\ 1\le j\le k}} \pmod q$$

is maximal. [If $k \ge m = (n+1)^r$ and $\bar{x}_1,\dots,\bar{x}_r$ satisfy the regularity condition, then rank M (mod q) = m.]

Lemma 3 (on the number of solutions of a complete system of congruences). Let p be a prime and T be the number of solutions of the system of congruences:

$$\sum_{j=1}^{2m} (-1)^j\, x_{1,j}^{t_1}\cdots x_{r,j}^{t_r} \equiv 0 \pmod{p^{t_1+\dots+t_r}},\quad 0\le t_1,\dots,t_r\le n,$$

$B \le x_{s,j} < B + p^{rn}$, $s = 1,\ldots,r$, $j = 1,\ldots,2m$. Suppose that $\bar{x}_j$, $j = 2,4,\ldots,2m$ satisfy the regularity condition mod p. Then

$$T \le m!\, p^{2mr^2n - \frac{rmn}{2}} .$$

Fundamental Lemma. Suppose that $n \ge 2$, $k \ge 2m$, $P \ge 1$. Then $\exists p \in [P^{1/nr}, 2P^{1/nr}]$ such that

$$J(P;n,k,r) \le 2(rn)^2 \binom{k}{m}^2 p^{2r(k-m)} TJ(P_1;n,k-m,r) + 2J_2$$

$$\le 2k^{2m} p^{2mr^2n+2rk-\frac{rmn}{2}-2rm} J(P_1;n,k-m,r) + \frac{1}{8}(2^r rn)^{2rkn} P^{2rk-k},$$

where $P_1 = Pp^{-1} + 1$.

Theorem 3 (Mean value theorem). If $0 \le \tau \in \mathbb{Z}$, $k \ge m\tau$, $P \ge 1$, then

$$J \le k^{2m\tau} 4^{mr^2n\tau} (nr)^{2nr\Delta(\tau)} P^{2rk-\frac{rmn}{2}+\delta(\tau)} ,$$

where

$$\delta(\tau) = \frac{rmn}{2}\left(1 - \frac{1}{rn}\right)^{\tau}, \quad \Delta(\tau) = \frac{rmn}{2} - \delta(\tau).$$

Chap. II Estimation of m.t.s.

§1. Lemma 2 (on multiple trigonometrical integrals). Suppose that $\alpha(0,\ldots,0) = 0$ and let $\alpha = \max\limits_{0\le t_1,\ldots,t_r\le n} |\alpha(t_1,\ldots,t_r)|$. Then

$$|I_r| = \left|\int_0^1 \cdots \int_0^1 \exp(2\pi i F(x_1,\ldots,x_r)) dx_1\cdots dx_r\right| \le \min\{1,\ 32^r \alpha^{-1/n} \cdot (\log(\alpha+1)+2)^{r-1}\} ,$$

the estimate being precise.

Lemma 8 a) (on complete m.t.s.). Suppose that $F(x_1,\ldots,x_r)$

$\in \mathbb{Z}[x_1,\dots,x_r]$, $(\alpha(0,\dots,1),\dots,\alpha(n,\dots,n),q) = 1$, $\alpha(0,\dots,0) = 0$.

Then
$$\left|S\left(\frac{F(x_1,\dots,x_r)}{q}\right)\right| = \left|\sum_{x_1=1}^{q}\cdots\sum_{x_r=1}^{q}\exp\left(2\pi i\frac{F(x_1,\dots,x_r)}{q}\right)\right|$$
$$\le (5n^{2n})^{r\omega(q)}(d(q))^{r-1}q^{r-\frac{1}{n}}.$$

§2. The simplest estimate (i.e. the one depending on approximation of a coefficient by a rational number, see [1]).

Theorem 1. If $0 \le t_1,\dots,t_r \le n$, $t_1+\cdots+t_r \ge 2$ such that
$$\alpha(t_1,\dots,t_r) = \frac{a}{q} + \frac{\theta}{q^2},\ (a,q) = 1,\ |\theta| \le 1,\ 1 < q < P^{t_1+\cdots+t_r},$$
then
$$|S_u(A)| < 3^{10r^2n+3\log(\omega/\rho)}u^{3\rho/\omega}P^{r-\rho},$$
where ω is defined by the relations $q = P^{\omega}$, $\omega = 1$, $q = P^{t_1+\cdots+t_r-\omega}$, according as $1<q<P$, $P \le q \le P^{t_1+\cdots+t_r-1}$, $P^{t_1+\cdots+t_r-1} < q < P^{t_1+\cdots+t_r}$, and
$$\rho = \omega/(5mrn\log(\frac{3rnm}{2\omega})).$$

§3. A general estimate for m.t.s.

The points $0 \le \alpha(t_1,\dots,t_r) \le 1$, $t_1+\cdots+t_r \ge 1$ are divided into two classes depending on their approximation by rational fractions:
$$\Omega = \Omega_1 \cup \Omega_2,\quad \text{mes}\ \Omega_1 = O(P^{-\frac{rnm}{2}+1+\frac{2}{m}}),\quad \text{mes}\ \Omega_2 = 1 - \text{mes}\ \Omega_1.$$

On Ω_1 the estimate for $|S(A)|$ is obtained, (which is, in most cases, unimprovable,) using the estimates for m.t.s. from §1 and a generalization of van der Corput's lemma

(Lemma 15).

On Ω_2 the following uniform estimate is obtained on the basis of Lemma 14 on the intersection multiplicity of domains:

$$|S(A)| \le e^{cr^2 n} P^{r-\rho}, \quad \rho = \frac{c_1}{rnm \log(rnm)}.$$

3. General estimates

Theorem 2. Suppose that

$$\alpha(t_1,\dots,t_r) = \frac{a(t_1,\dots,t_r)}{q(t_1,\dots,t_r)} + z(t_1,\dots,t_r), \quad (a(t_1,\dots,t_r),$$

$$q(t_1,\dots,t_r)) = 1, \quad 0 \le t_1,\dots,t_r \le n, \quad Q = \underset{1 \le t_1+\cdots+t_r}{\mathrm{LCM}} q(t_1,\dots,t_r).$$

$$\alpha(t_1,\dots,t_r) \in \Omega_1 \overset{\text{def}}{\Longleftrightarrow} Q < P^{0.1}, \ |z(t_1,\dots,t_r)| < P^{-(t_1+\cdots+t_r)+\frac{1}{m}},$$

$\alpha(t_1,\dots,t_r) \in \Omega_2 \overset{\text{def}}{\Longleftrightarrow} \alpha(t_1,\dots,t_r) \notin \Omega_1$. Then on Ω_1,

$$|S| \le (10n^{2n})^{r\omega(Q)} (d(Q))^{r-1} P^r Q^{-1/n}.$$

Moreover, putting $\delta(t_1,\dots,t_r) = z(t_1,\dots,t_r) P^{t_1+\cdots+t_r}$,

$\delta = \max\limits_{0 \le t_1,\dots,t_r \le 1} \delta(t_1,\dots,t_r)$, $\delta(0,\dots,0) = 0$, then on Ω_1,

$$|S| \le 32^r (10n^{2n})^{r\omega(Q)} (d(Q))^{r-1} P^r (Q\delta)^{-1/n} (\log(1+\delta)+1)^{r-1}.$$

Theorem 3 is used in §§1 & 2 in Chap. III.

4. An estimate of a complete m.t.s. by the mean value theorem (Theorem 3 in Chap. I).

Theorem 4. Let $q(0,\dots,0),\dots,q(n,\dots,n) \in \mathbb{N}$ and Q be their LCM. Suppose that

$(a(t_1,\ldots,t_r), q(t_1,\ldots,t_r)) = 1,\ 0 \le t_1,\ldots,t_r \le 1,\ t_1 + \cdots + t_r \ge 1,$

and define

$$U := \sum_{x_1=1}^{Q} \cdots \sum_{x_r=1}^{Q} \exp\Big(2\pi i \sum_{\substack{t_1=0 \\ t_1+\cdots+t_r\ge 1}}^{n} \cdots \sum_{t_r=0}^{n} \frac{a(t_1,\ldots,t_r)}{q(t_1,\ldots,t_r)} x_1^{t_1}\cdots x_r^{t_r}\Big).$$

Then $\exists c,\ c_1 > 0$ such that

$$|U| \le e^{cr^2 n} Q^{r-\rho}, \quad \rho = \frac{c_1}{rnm \log(rnm)}.$$

Chap. III Applications of the theory of m.t.s.

§1. Precise upper bounds for the number of solutions of complete and incomplete systems of Diphantine eqns.

1. Problem setting. To obtain an estimate for J, asymptotically precise with true number k of summands w.r.t. n and r, i.e. to obtain an estimate for J of the form

$$c_1 D(P) \le J \le c_2 D(P), \tag{1}$$

where c_1 and c_2 depend only on n, r & k, $D(P) \to \infty$ as $P \to \infty$ and (1) holds for $k \ge k_0 = k_0(n,r)$; moreover, the bound for k for which (1) holds, i.e. the quantity k_0 should essentially decreases, which means that for any fixed $\varepsilon > 0$, (1) does not hold for $k < k_0^{1-\varepsilon}$.

2. Theorems.

Theorem 1. $\exists c,\ c_1 > 0$ such that for $n \ge 2$, $P \ge 1$, $k > c_1 rnm \log(rnm)$, we have

$$J = J(P;n,k,r) \le e^{cr^3n^2m \log(rnm)} P^{2kr-\frac{rnm}{2}},$$

where from now on $\alpha(0,\ldots,0) = 0$.

Theorem 2 (On an incomplete system of Diphantine eqns). Under the same conditions as in Theorem 1, we have

$$J_H = J_H(P;n,k,r) = \int\cdots\int_{\Omega_H} |S_H(A)|^{2k} dA \le e^{cr^3n^2m \log(rnm)} P^{2kr-\omega},$$

where

$$S_H(A) = \sum_{x_1=1}^{P} \cdots \sum_{x_r=1}^{P} \exp(2\pi i g(x_1,\ldots,x_r)),$$

$$g(x_1,\ldots,x_r) = \alpha(s_1,\ldots,s_r)x_1^{s_1}\cdots x_r^{s_r} + \cdots + \alpha(t_1,\ldots,t_r)x_1^{t_1}\cdots x_r^{t_r}$$

$$+ \cdots + \alpha(u_1,\ldots,u_r)x_1^{u_1}\cdots x_r^{u_r},$$

Ω_H : the m_H-dimensional unit cube, m_H = the number of non-zero α's and

$$\omega = (s_1+\cdots+s_r) + \cdots + (t_1+\cdots+t_r) + \cdots + (u_1+\cdots+u_r).$$

That is, Theorem 2 gives the precise upper bound for the mean value of degree $2k$ of the modulus of r-multiple t.s. with polynomials in their exponent, part of whose coefficients are identically zero. (The suffix "H" is an abbreviation of "He" —— negation in Russian.)

3. Estimates from below

Theorem 3 (which shows that the parameter k in Theorem 1 has the true order). Let k_0 be such that for all $k \ge k_0$ (1) holds, i.e.

$$J_r = J(P;n,k,r) \le c(n,k,r)P^{2kr-\Delta_r},$$

where

$$\Delta_r = \frac{rnm}{2} = \frac{rn(n+1)^r}{2}.$$

Then $\exists c_0, c > 0$ such that

$$c_0\Delta_r < k_0 < c\Delta_r \log \Delta_r .$$

§2. Asymptotic formulas for the number of solutions of a complete and an incomplete systems of Diophantine eqns.

2. Asymptotic formulas

Theorem 4. Let $k_0 = crnm \log(rnm)$, $c, c_1 > 0$. Then for $k \ge k_0$ we have

$$J = \sigma\theta P^{2kr-\frac{rnm}{2}} + O(P^{2kr-\frac{rnm}{2}-\frac{c_1}{\log(rnm)}}),$$

where σ and θ are resp. the singular series and the singular integral defined by

$$\theta = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left|\int_0^1\cdots\int_0^1 \exp\{2\pi i F(x_1,\ldots,x_r)\}\, dx_1\cdots dx_r\right|^{2k} d\alpha(n,\ldots,n) \cdots d\alpha(0,\ldots,0,1),$$

$$\sigma = \sum_{q(n,\ldots,n)=1}^{+\infty}\cdots\sum_{q(0,\ldots,0,1)=1}^{+\infty}\ \sum_{\substack{a(n,\ldots,n)=1\\(a(n,\ldots,n),q(n,\ldots,n))=1}}^{q(n,\ldots,n)}\cdots\sum_{(a(0,\ldots,0,1),q(0,\ldots,0,1))=1}^{q(0,\ldots,0,1)}\left|q^{-r}\sum_{x_1=1}^{q}\cdots\sum_{x_r=1}^{q}\exp(2\pi i\sum_{t_1=0}^{n}\cdots\sum_{t_r=0}^{n}\frac{a(t_1,\ldots,t_r)}{q(t_1,\ldots,t_r)}x_1^{t_1}\cdots x_r^{t_r})\right|^{2k},$$

$$q = \prod_{t_1=0}^{n} \cdots \prod_{t_r=0}^{n} q(t_1,\dots,t_r),\ q(0,\dots,0) = 1.$$

Theorem 5. Let $k > k_0 = crnm\log(rnm)$, $c > 0$. Then $\exists\, c_1 > 0$ such that

$$J_H = \sigma_H \theta_H P^{2kr-\omega} + O(P^{2kr-\omega-\frac{c_1}{\log(rnm)}}),$$

where σ_H and θ_H are defined from σ and θ by restricting them to non-zero $\alpha(t_1,\dots,t_r)$.

Note that J_H is the number of the following incomplete system of Diophantine eqns:

$$\begin{cases} \sum_{j=1}^{2k} (-1)^j x_{1,j}^{s_1} \cdots x_{r,j}^{s_r} = 0, \\ \dots\dots\dots \\ \sum_{j=1}^{2k} (-1)^j x_{1,j}^{t_1} \cdots x_{r,j}^{t_r} = 0, \\ \dots\dots \\ \sum_{j=1}^{2k} (-1)^j x_{1,j}^{u_1} \cdots x_{r,j}^{u_r} = 0. \end{cases}$$

Theorem 6. For $s \le n$ let

$$H = H(P;n,k,r) = \int\cdots\int_{\Omega} \left| \sum_{\substack{x_1=1 \\ x_1^s+\cdots+x_r^s \le P^s}}^{P} \cdots \sum_{x_r=1}^{P} \exp(2\pi i F(x_1,\dots,x_r)) \right|^{2k} dA.$$

Then with the same k_0, σ, $F(x_1,\dots,x_r)$ as in Theorem 4, we have for $k \ge k_0$

$$H(P;n,k,r) = \sigma\theta_1 P^{2kr-\frac{rnm}{2}} + O(P^{2kr-\frac{rnm}{2}-\frac{c_1}{\log(rnm)}}),$$